JOURNAL OF COMPUTATIONAL PHYSICS **61**, 345–348 (1985)

Note

The Core Spreading Vortex Method Approximates the Wrong Equation

In the (nonrandom) vortex blob method, solutions to Euler's equations for incompressible, inviscid fluid flow are approximated by the motion of finitely many cores of vorticity of fixed shape. Two variations of the inviscid algorithm have been proposed for the computation of solutions to the Navier–Stokes equations. These are Chorin's random vortex method and what we shall call the core spreading method (see Leonard's survey [4] for a discussion of both of these methods and a list of references). In Chorin's method, vortex cores of fixed shape make random jumps for the simulation of diffusion. In the core spreading method, on the other hand, the cores of vorticity are Gaussian functions spreading in time as exact solutions of the heat equation, and randomness is eliminated.

The purpose of this note is to point out that the core spreading algorithm is physically wrong and, indeed, converges to a system of equations different from the Navier–Stokes equations. In the core spreading algorithm, vorticity is correctly diffused, but incorrectly convected, even in the limit of infinitely many vortices. We restrict our attention to the simplest case, that is, 2-dimensional flow without boundaries. We take the viscosity to have value one and denote by η the vorticity at time $t=0$, which we assume to be smooth and of compact support.

In the core spreading method (as in the inviscid vortex blob method) the trajectories $x_i(t)$ of a finite number of fluid particles are calculated by the integration of a system of autonomous ordinary differential equations. In these differential equations, the velocities dx_i/dt are computed from the vorticity distribution determined by the system of all the particles. Let ϕ_t denote the shape of each vortex core at time t, so that $\phi_t(x)=\phi(x, t)$, where $\phi(x, t)$ is the solution of the heat equation with initial condition $\phi_0(x)$. Thus, $\phi_t = G_t * \phi_0$, where $G_t(x)=(4\pi t)^{-1}\exp(-|x|^2/4t)$ is the heat kernel, and where $*$ is the convolution operator, so that $G_t * \phi_0(x)=\int G_t(x-x')\,\phi_0(x')\,dx'$. The vorticity distribution from which velocities at time t are calculated is assumed to have the form

$$\zeta(x, t)=\sum_i \phi_t(x-x_i(t))\,\eta_i,$$

where η_i is typically either the total vorticity in a small region about $\alpha_i = x_i(0)$, or, when the initial positions α_i are the nodes of a grid of mesh width h, $\eta_i=\eta(\alpha_i)\cdot h^2$;

0021-9991/85 $3.00

the summation is over all the particles. Set $K(x)=\nabla\times(-\log)(x)=(-x_2, x_1)/|x|^2$ and $\tilde{K}_t=K*\phi_t$. The velocity field v whose curl $\nabla\times v$ is ζ is given by

$$v(x, t)=K*\zeta(x, t)=\sum_i \tilde{K}_t(x-x_i(t))\,\eta_i.$$

The core spreading method consists of the system of equations

$$x_i(0)=\alpha_i, \tag{1}$$

$$\frac{dx_i}{dt}(t)=\sum_j \tilde{K}_t(x_i(t)-x_j(t))\,\eta_j. \tag{2}$$

Set $K_t=K*G_t$, and consider the Lagrangian system of equations

$$\tilde{\Phi}(\alpha, 0)=\alpha, \tag{3}$$

$$\frac{\partial\tilde{\Phi}}{\partial t}(\alpha, t)=\int K_t(\tilde{\Phi}(\alpha, t)-\tilde{\Phi}(\beta, t))\,\eta(\beta)\,d\beta. \tag{4}$$

Observe that $\tilde{K}_t=K*\phi_t=K*G_t*\phi_0=K_t*\phi_0$. Thus, Eqs. (1)–(2) and (3)–(4) differ from the inviscid vortex blob method (with core function ϕ_0) and the flow map formulation of Euler's equations, respectively, only in the replacement of K by K_t. In fact, if ϕ_0 is radially symmetric and sufficiently smooth, with $\int_{\mathbb{R}^2}\phi_0=1$, the α_i are chosen to be the nodes of a grid of mesh width h, $\eta_i=\eta(\alpha_i)\cdot h^2$, and ϕ_0 tends to the Dirac distribution while h goes to zero, then the proof of the convergence theorem of Beale and Majda [3] (or see [1]) can easily be adapted to the system (1)–(2) to show its convergence to (3)–(4). We now compare (3)–(4) to the Navier–Stokes equations.

Define

$$\tilde{u}(x, t)=\int K_t(x-\tilde{\Phi}(\beta, t))\,\eta(\beta)\,d\beta. \tag{5}$$

Thus, Eq. (4) can be written in the form $\partial\tilde{\Phi}/\partial t(\alpha, t)=\tilde{u}(\tilde{\Phi}(\alpha, t), t)$. Let ξ denote the passive transport of η by $\tilde{\Phi}$; then $\xi(\tilde{\Phi}(\alpha, t), t)=\eta(\alpha)$, and so $\partial\xi/\partial t+(\tilde{u}\cdot\nabla)\,\xi=0$. The vorticity corresponding to the system (3)–(4) is the function $\tilde{\omega}=\nabla\times\tilde{u}$. Since, by a change of variables in the integral (5), $\tilde{u}=K_t*\xi=K*(G_t*\xi)$, it follows that $\tilde{\omega}=\nabla\times\tilde{u}=G_t*\xi$.

Whereas the Navier–Stokes vorticity satisfies

$$\frac{\partial\omega}{\partial t}=\nabla^2\omega-(u\cdot\nabla)\,\omega,$$

the vorticity $\tilde{\omega}$ satisfies

$$\frac{\partial \tilde{\omega}}{\partial t} = \frac{\partial}{\partial t}(G_t * \xi) = \nabla^2 (G_t * \xi) + G_t * \frac{\partial \xi}{\partial t} = \nabla^2 \tilde{\omega} - G_t * (\tilde{u} \cdot \nabla \xi). \tag{6}$$

Taking second derivatives, we have

$$\frac{\partial^2 \omega}{\partial t^2} = \nabla^2 \frac{\partial \omega}{\partial t} - \left(\frac{\partial u}{\partial t} \cdot \nabla\right) \omega - (u \cdot \nabla)\, \nabla^2 \omega + (u \cdot \nabla)((u \cdot \nabla)\, \omega),$$

while

$$\frac{\partial^2 \tilde{\omega}}{\partial t^2} = \nabla^2 \frac{\partial \tilde{\omega}}{\partial t} - \nabla^2 (\tilde{u} \cdot \nabla \xi) - \frac{\partial \tilde{u}}{\partial t} \cdot \nabla \xi + (\tilde{u} \cdot \nabla)((\tilde{u} \cdot \nabla)\, \xi).$$

Since at time $t = 0$,

$$\frac{\partial \tilde{\omega}}{\partial t} = \frac{\partial \omega}{\partial t} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial t} = \frac{\partial u}{\partial t},$$

we have, setting $u_0(x) = u(x, 0) = K * \eta(x)$,

$$\frac{\partial^2 \tilde{\omega}}{\partial t^2} - \frac{\partial^2 \omega}{\partial t^2} = (u_0 \cdot \nabla)\, \nabla^2 \eta - \nabla^2((u_0 \cdot \nabla)\, \eta)$$

at time $t = 0$. The right-hand side is in general nonzero, and so ω and $\tilde{\omega}$ are not the same functions. Observe, however, that in the radially symmetric case, when core spreading happens to converge to the correct equations, both terms on the right-hand side are zero.

It is interesting to compare the discussion given here with the results of [2]. Beale and Majda show that a convergent approximation to the Navier–Stokes equations in free space is obtained by a splitting procedure in which, at each time step, Euler's equations are solved exactly, and then the heat equation is solved exactly. In the core spreading method, the Euler part of the splitting is incorrectly solved, for the vorticity is convected, not by the local velocity field, but by an averaged velocity, as is clear from (6). Chorin's random vortex method, on the other hand, can be seen to be an approximation to the correct splitting procedure: at each time step, Euler's equations are solved and then the heat equation is approximated by particle diffusion. In fact, Marchioro and Pulvirenti [5] have shown that solutions of the stochastic differential equation, which is the continuous-time version of the random vortex method, converge to solutions of the Navier–Stokes equations.

References

1. C. Anderson and C. Greengard, *SIAM J. Numer. Anal.* **22** (1985), 413.
2. J. T. Beale and A. Majda, *Math. Comput.* **37** (1981), 243.
3. J. T. Beale and A. Majda, *Math. Comput.* **39** (1982), 29.
4. A. Leonard, *J. Comput. Phys.* **37** (1980), 289.
5. C. Marchioro and M. Pulvirenti, *Commun. Math. Phys.* **84** (1982), 483.

Received: October 25, 1984

Claude Greengard

Courant Institute of Mathematical Sciences,
New York University,
251 Mercer Street, New York, New York 10012